LETTERS TO THE EDITOR

CAN MOTION ON A TRACK MIMIC A FORCED, DAMPED NONLINEAR OSCILLATOR?

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1. INTRODUCTION

In some earlier papers [1–3] the author considered the correspondence between frictionless motion of a particle on a curved track under the influence of gravity and one-dimensional oscillator motion due to a potential. Systems for which the motion on a track, *projected onto the horizontal axis*, was exactly the same as a one-dimensional motion due to a potential were termed isodynamical. The potential isodynamical to any given track shape was found in reference [1]; and the tracks isodynamical to any given potential were treated in reference [2].

The above work dealt with undamped (frictionless) and unforced (undriven) motion. Insofar as forced oscillations exhibit a much richer range of phenomena, it is therefore of interest to know further whether a track could be constructed, for which the projected motion is isodynamical to a forced, possibly damped, non-linear oscillator. That is, can a mechanical system still exactly mimic the one-dimensional oscillator dynamics?

Gottwald *et al.* [4] attempted an experimental track realization of the Duffing equation, but they made approximations, and actually dealt with mimicry of general phenomena rather than exact mimicry (isodynamism) of the oscillator equation. At about the same time, Shaw and Haddow [5] investigated the dynamics of "roller-coaster" experimental models as analogue devices for non-linear oscillators, including the effects of friction and excitation by horizontal support motion. As emphasized by them, they chose the arc-length co-ordinate equation as the dynamical equation required to have the same form as the oscillator equation. Their example dealing with the forced situation also involved some approximations. In a recent book, Virgin [6] has extended these investigations.

If friction is included, then a common qualitative way of thinking (e.g. Alligood *et al.* [7, p. 378]) is to include a linear damping term in an oscillator equation, and imagine a ball rolling up and down a potential well of the same shape as obtained from the energy equation and losing energy because of friction. However, this correspondence is not expected to be exact.

In this paper, we attempt to include (viscous) friction, proportional to velocity, and driven motion. The differential equations for all these situations are derived. It is shown that isodynamical motion, as we have defined it above, is not possible when the damping is included; nor is it possible for motion forced by a horizontally driven, oscillating track.

2. DAMPED MOTION

2.1. DAMPED, NON-LINEAR OSCILLATOR

For a particle, with mass m, whose motion in one dimension is due to a potential $v(x)$ in the presence of viscous damping, with coefficient γ, the equation of motion is

$$m\,\ddot{x} + \gamma\dot{x} + \mathrm{d}v/\mathrm{d}x = 0. \tag{2.1}$$

Using $\ddot{x} = (\frac{1}{2})\,(\mathrm{d}/\mathrm{d}x)(\dot{x}^2)$ and $\mathrm{d}x = \dot{x}\,\mathrm{d}t$, this may be integrated to give

$$\dot{x}^2 + (2/m)v(x) = -2(\gamma/m)\int \dot{x}^2\,\mathrm{d}t. \tag{2.2}$$

2.2. MOTION ON A TRACK WITH FRICTION

A particle, say a bead, with mass M, slides, under the influence of gravity (constant g), along a track which lies in a vertical plane and which has equation $Y = Y(X)$, where X is the horizontal co-ordinate and Y the vertical co-ordinate. If the friction were of the "dry" type, proportional to the normal reaction, then the equations of motion would not contain a velocity-type term such as that in the usual damped oscillator equation (2.1). Therefore, for the possibility of motion isodynamical with the damped oscillator, the friction associated with motion along the track must evidently be of the "viscous" type, proportional to tangential velocity. The friction forces in the X and Y directions are then simply $-\Gamma\dot{X}$ and $-\Gamma\dot{Y}$ respectively (Γ constant).

The vertical acceleration can be expressed in terms involving the slope Y' by

$$\ddot{Y} \equiv Y'\ddot{X} + Y''(\dot{X})^2. \tag{2.3}$$

Resolution of forces and elimination of the normal reaction leads eventually to the equation for motion projected onto the horizontal axis:

$$\ddot{X} = -[(g + Y''\dot{X}^2 + (\Gamma/M)Y'\dot{X})Y' + (\Gamma/M)\dot{X}]/[1 + (Y')^2]. \tag{2.4}$$

(Cf. equation (2.3) in reference [1] when $\Gamma = 0$.) This can be integrated to give

$$\dot{X}^2 + 2gY/[1 + (Y')^2] = -2(\Gamma/M)[\int \dot{X}^2[1 + (Y')^2]\,\mathrm{d}t]/[1 + (Y')^2]. \tag{2.5}$$

Comparison of the right-hand side of equation (2.2) with that of (2.5) shows that they cannot be of similar form, except for the trivial case of constant slope. (Actually, piecewise constant slope, such as a V-shape, could be accommodated—cf. the Appendix of reference [1] for the undamped case—but this will not be considered further here as it is not a smooth shape.)

It is thus concluded that, unlike the undamped case, no isodynamical systems of potential motion and curved track can be found when friction is present.

3. FORCED MOTION

3.1. FORCED NON-LINEAR OSCILLATOR

The equation of motion in one dimension for a particle of mass m due to a potential $v(x)$ and subject to a driving force written as $m\hat{d}(t)$ is

$$m\ddot{x} + \mathrm{d}v/\mathrm{d}x = m\hat{d}(t), \tag{3.1}$$

which may be integrated once to give

$$\dot{x}^2 + (2/m)v = 2\int \dot{x}\hat{d}(t)\,dt. \tag{3.2}$$

3.2. MOTION ON A DRIVEN TRACK

The track (regarded as a rigid shape) may be subjected to a prescribed horizontal displacement, i.e., shaken, as in reference [4] for harmonic excitation or reference [5] more generally. (In view of the result of section 2 above, the motion is taken to be undamped.) The track shape is now denoted by $F(X)$, and the track undergoes the given time-dependent displacement $D(t)$. At time t, the vertical co-ordinate of the particle (bead of mass M) is therefore given by

$$Y(X) = F[\Xi(t)], \quad \Xi(t) \equiv X - D(t). \tag{3.3}$$

The slope at X is therefore given by F', where the prime now denotes differentiation with respect to Ξ.

For the possibility of isodynamical motion, a horizontal co-ordinate change of independent variable is made, from X to Ξ, so that, when $F = 0$, the equation for $\ddot{\Xi}$ looks like the equation for $\ddot{x}$ when $v = 0$. The equation of motion relative to the moving frame, using equation (3.3), may then be shown to be

$$\ddot{\Xi} = -[(g + F''\dot{\Xi}^2)F' + \ddot{D}]/[1 + (F')^2]. \tag{3.4}$$

(When $D = 0$, this agrees with equation (2.3) of reference [1]. For D non-zero and $F = 0$, this can look like equation (3.1) with $v = 0$.) Equation (3.4) can be integrated once to give

$$\dot{\Xi}^2 + \frac{2gF(\Xi)}{1 + (F')^2} = \frac{2\int \dot{\Xi}\ddot{D}\,dt}{1 + (F')^2}. \tag{3.5}$$

Comparison with the right-hand side of equation (3.2) shows that, once again, because the denominator in the last term of equation (3.5) is a function of co-ordinate, this projected motion (relative to the driven track frame) cannot be isodynamical with an oscillator in the forced case.

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